Submanifolds with nonparallel first normal bundle revisited

Marcos Dajczer and Ruy Tojeiro

Abstract

In this paper, we analyze the geometric structure of an Euclidean submanifold whose osculating spaces form a nonconstant family of proper subspaces of the same dimension. We prove that if the rate of change of the osculating spaces is small, then the submanifold must be a (submanifold of a) ruled submanifold of a very special type. We also give a sharp estimate of the dimension of the rulings.

The osculating space at a point of an Euclidean submanifold M^n is the subspace in Euclidean space spanned by the tangent and curvature vectors of all smooth curves in M^n through that point. It is an elementary fact that if all osculating spaces along M^n coincide with a fixed subspace H, then M^n is contained in an affine subspace parallel to H. It is a natural problem to study for which submanifolds the osculating spaces form a nonconstant family of proper subspaces of the same dimension along M^n.

In this paper we show that if the rate of change of the osculating spaces is small, in a sense to be made precise below, then the submanifold must be contained in a ruled submanifold of a very special type.

Let $f \colon M^n \to \mathbb{R}^N$ denote an isometric immersion of an n-dimensional connected Riemannian manifold into Euclidean space. The *first normal space* of f at $x \in M^n$ is the normal subspace $N_1^f(x) \subset N_x M$ spanned by its second fundamental form α_f, i.e.,

$$N_1^f(x) = \operatorname{span}\{\alpha_f(X,Y) : X,Y \in T_x M\}.$$

The osculating space of f at $x \in M^n$ is $f_* T_x M \oplus N_1^f(x)$. Hence, the condition that the osculating spaces of f be a constant proper subspace along M^n is equivalent to the first normal spaces forming a proper normal subbundle N_1^f that is parallel in the normal connection; see [1] or [7]. Then f reduces codimension to $p = \operatorname{rank} N_1^f$, i.e., it can be seen as a substantial isometric immersion into an affine subspace $\mathbb{R}^{n+p}$ of $\mathbb{R}^N$.

A rather simple argument shows that N_1^f must be parallel in the normal connection if $p < n$ and the s-nullities ν_s of f satisfy at any $x \in M^n$ that

$$\nu_s(x) < n - s \tag{1}$$

for all $1 \leq s \leq p$; see [1], [4] or (13) below. Recall that

$$\nu_s(x) = \max_{U^s \subset N_1^f(x)} \dim \mathcal{N}(\alpha_{U^s})$$

where $U^s \subset N_1^f(x)$ is any s-dimensional vector subspace and

$$\mathcal{N}(\alpha_U) = \{Y \in TM : \alpha_U(Y, X) = 0 : X \in TM\}$$

for $\alpha_U = \pi_U \circ \alpha_f$ and $\pi_U = N_1^f \to U$ the orthogonal projection. Notice that $\nu_p(x)$ is the standard *index of relative nullity* $\nu^f(x) = \dim \mathcal{N}(\alpha_f(x))$, i.e., the dimension of the *relative nullity* subspace of f at $x \in M^n$.

In order to measure to what extent the first normal bundle N_1^f fails to be parallel, we consider its subbundle $\mathcal{S}$ spanned by the projections $(\nabla_X^\perp \mu)_{N_1}$ onto N_1^f of derivatives of sections of its orthogonal complement $N_1^\perp$. The greater the rank of $\mathcal{S}$, the more N_1^f is nonparallel.

If $\mathcal{S}$ coincides with N_1^f and has rank $s = p \leq 6$, it turns out that condition (1) fails for the relative nullity, i.e., $\nu^f \geq n - p > 0$ at any point. This has strong well-known geometric consequences, namely, the submanifold carries a ν^f-dimensional totally geodesic foliation whose leaves are open subsets of affine subspaces in $\mathbb{R}^N$.

Our main result is that there is a single class of submanifolds for which $\mathcal{S}$ is a proper subbundle of rank $s \leq 6$, any other example being a submanifold of an element of this class. These are ruled submanifolds, with rulings of dimension at least $n - s$, for which $\mathcal{S}$ is constant in the ambient space along the rulings. In particular, the rulings belong to the kernel of $\alpha_{\mathcal{S}}$, and therefore condition (1) is violated for s. Examples of such submanifolds, showing that the preceding estimate on the dimension of the rulings is sharp, are constructed in the last section.

As discussed in the next section, the results of this paper generalize the ones in [5] for $p \leq 3$. We point out that, although stated for submanifolds of Euclidean space, they can easily be extended to ambient spaces of constant sectional curvature.

1 The result

In this section, we first give a precise statement of our main result. Then, we give an application and discuss some particular cases of it.

Let $f \colon M^n \to \mathbb{R}^N$ denote a locally substantial isometric immersion of a connected Riemannian manifold, i.e., there is no open subset $U \subset M^n$ such that $f(U)$ is contained in a proper affine subspace of $\mathbb{R}^N$. Assume that f is 1-regular, i.e., the first normal spaces $N_1^f(x)$ have constant dimension p. Thus, these subspaces form a vector subbundle $N_1 = N_1^f$ of the normal bundle $N_f M$ which we assume to be proper, i.e., $p < N - n$.

Assume $p < n$ and let $\phi \colon N_1^\perp \oplus TM \to N_1$ be the tensor defined by

$$\phi(\mu, X) = (\nabla_X^\perp \mu)_{N_1}.$$

We say that f has *nonparallel first normal bundle at* $x \in M^n$ if $\phi(x) \neq 0$, i.e., the dimension $s(x)$ of the vector subspace $\mathcal{S}(x) \subset N_1^f(x)$ given by

$$\mathcal{S}(x) = \mathrm{span}\{\phi(\mu, X) : \mu \in N_1^\perp(x) \text{ and } X \in T_x M\}$$

is nonzero. Thus, along each connected component of the open dense subset of M^n where $s(x) = s$ is constant, the vector subspaces $\mathcal{S}(x)$ form a vector subbundle $\mathcal{S} = \mathcal{S}^s$ of N_1^f.

In the following statement, that an isometric immersion $F \colon N^m \to \mathbb{R}^N$, $m > n$, is an *extension* of the isometric immersion $f \colon M^n \to \mathbb{R}^N$ means that there exists an isometric embedding $i \colon M^n \to N^m$ such that $f = F \circ i$. Also, by f being D^d-ruled we understand that D^d is a d-dimensional integrable distribution in M^n whose leaves are (mapped by f into) open subsets of affine subspaces in the ambient space.

Theorem 1. *Let $f \colon M^n \to \mathbb{R}^N$ be a 1-regular locally substantial isometric immersion such that $s(x)$ has a constant value $0 < s < n$. Assume further that $s \leq 6$. Then, either*

   (i) $s = p$ and f has index of relative nullity $\nu_f \geq n - p$, or

   (ii) $1 = s < p$ and f has an extension $F \colon N^m \to \mathbb{R}^N$, $m = n + p - 1$, such that $\nu_F = m - 1$ and N_1^F is nonparallel of rank one, or

  (iii) $1 < s < p$ and there is an open dense subset of M^n with connected components U_k, $s \leq k \leq p$, such that

      (a) $f|_{U_p}$ is D^d-ruled with $d \geq n - s$, where $D = \mathcal{N}(\alpha_\mathcal{S})$ and $\mathcal{S}$ is constant along D in $\mathbb{R}^N$, and

      (b) $f|_{U_k}$ for $k < p$ extends to $F \colon N^m \to \mathbb{R}^N$, with $m = n + p - k$ and $\mathrm{rank}\, N_1^F = k$, which has $\nu_F \geq m - k$ if $s = k$ and is Δ^q-ruled, $q \geq m - s$, if $s < k$.

   Moreover, we have that $U_k = \emptyset$ if $s = 2$ and $k = 5, 6$.

For a ruled Euclidean submanifold, it is easily seen that for any vector X tangent to a ruling the Ricci curvature satisfies $\mathrm{Ric}(X) \leq 0$, with equality if and only if X belongs to the relative nullity subspace. Thus, we have the following immediate consequence of Theorem 1.

Corollary 2. *Under the assumptions of Theorem 1, cases (i) and $(iii) - (a)$ cannot occur if $\mathrm{Ric}_M > 0$. If $\mathrm{Ric}_M \geq 0$ then the foliation D in $(iii) - (a)$ is of relative nullity.*

To illustrate Theorem 1 we discuss next the cases $p = 1, 2$ and 3. Notice that these are the cases that were considered in [4].

Example 3. The case $p = 1$. In this case f satisfies $\nu^f = n - 1$. Thus M^n is flat.

Submanifolds as above can be easily described parametrically. For instance, consider the image under the normal exponential map of a parallel normal subbundle of the normal bundle of a curve with non-vanishing curvature; see also Theorem 1 in [5].

Example 4. The case $p = 2$. Here $(s, k) \in \{(2, 2), (1, 1)\}$ since $(1, 2)$ cannot occur. For simplicity, assume that only one value for (s, k) occurs globally. We have the following two possibilities:

(i) f satisfies $\nu_f = n - 2$ and $\mathcal{S} = N_1^f$.

(ii) f has an extension $F \colon N^{n+1} \to \mathbb{R}^N$ such that $\nu_F = n$ (N^{n+1} is flat) and N_1^F is nonparallel of rank one.

The submanifolds in (i) have been studied in [2] and [3], where a parametric classification was obtained in most cases.

Example 5. The case $p = 3$. Here $(s, k) \in \{(3, 3), (2, 3), (2, 2), (1, 1)\}$ since $(1, 2)$ and $(1, 3)$ cannot occur. Again, assume that only one value for (s, k) occurs globally. Then one of the following holds:

(i) f satisfies $\nu_f = n - 3$ and $\mathcal{S} = N_1^f$.

(ii) f is D^{n-2}-ruled and $\mathcal{S}$ has rank two and is constant along the rulings.

(iii) f has an extension $F \colon N^{n+1} \to \mathbb{R}^N$ with $\nu_F = n - 1$ and $N_1^F = \mathcal{S}$ of rank two.

(iv) f has an extension $F \colon N^{n+2} \to \mathbb{R}^N$ such that $\nu_F = n + 1$ (N^{n+2} is flat) and N_1^F is nonparallel of rank one.

Observe that F in (ii) of Example 4 and (iv) of Example 5 are as f in Example 3. Also, that F in (iii) of Example 5 is as f in (i) of Example 4.

2 A class of ruled extensions

In this section of independent interest, we find sufficient conditions for an Euclidean submanifold to admit a ruled extension carrying a normal subbundle that is constant along the rulings in the ambient space. We point out that a special case was already considered in [5].

Let $f \colon M^n \to \mathbb{R}^N$ be an isometric immersion whose normal bundle splits orthogonally and smoothly into two vector subbundles

$$N_f M = L \oplus P$$

where the rank ℓ of L satisfies $0 < \ell < N - n$. Assume that the subspaces

$$D^d(x) = \mathcal{N}(\alpha_P(x)) \subset T_x M$$

have constant dimension $d > 0$ on M^n and thus form a tangent subbundle $D \subset TM$. Assume also that P (hence L) is parallel along D in the normal connection. Notice that the latter condition is equivalent to the parallelism of P along D in $\mathbb{R}^N$.

Let $\gamma \colon E \oplus P \to E \oplus L$ be the tensor given by

$$\gamma(Y, \mu) = (\tilde{\nabla}_Y \mu)_{E \oplus L} = -A_\mu Y + (\nabla_Y^\perp \mu)_L, \tag{2}$$

where $\tilde{\nabla}$ is the connection in $\mathbb{R}^N$ and $E \subset TM$ is defined by the orthogonal splitting

$$TM = D \oplus E.$$

At $x \in M^n$, let $\Gamma(x) \subset E(x) \oplus L(x)$ be the subspace

$$\Gamma(x) = \mathrm{span}\{\gamma(Y, \mu) : Y \in E \text{ and } \mu \in P\}. \tag{3}$$

Since E is spanned by the vectors $A_\mu Y$ for $\mu \in P$ and $Y \in E$, it follows from (2) that

$$n - d = \dim E(x) \leq \dim \Gamma(x) \leq n - d + \ell. \tag{4}$$

Assume that $\dim \Gamma(x) = k$ is constant on M^n. Let $\pi \colon \Lambda \to M^n$ be the affine vector bundle of rank $r = n - d + \ell - k$ defined by the orthogonal splitting

$$\Gamma^k \oplus \Lambda^r = E^{n-d} \oplus L^\ell.$$

Lemma 6. *The distribution D is integrable and $\Lambda \cap TM = \{0\}$ holds.*

Proof: Take $\mu \in P$ and $Z, Y \in D$. Since P is parallel along D in $\mathbb{R}^N$, we have from

$$0 = \tilde{R}(Y, Z)\mu = \tilde{\nabla}_Y \tilde{\nabla}_Z \mu - \tilde{\nabla}_Z \tilde{\nabla}_Y \mu - \tilde{\nabla}_{[Y,Z]} \mu \tag{5}$$

that $\tilde{\nabla}_{[Y,Z]} \mu \in P$. Hence $A_\mu[Y, Z] = 0$, and thus D is integrable.

Take $Z \in \Lambda \cap TM$. Then $Z \in E$ and

$$0 = \langle Z, \tilde{\nabla}_X \mu \rangle = -\langle A_\mu Z, X \rangle$$

for any $\mu \in P$ and $X \in TM$. Thus $Z \in D$ and hence $Z = 0$. ∎

The affine subspaces $\Delta(x)$ defined by

$$\Delta(x) = D(x) \oplus \Lambda(x)$$

form an affine bundle over M^n of rank $d + r = n + \ell - k$.

Lemma 7. *The bundle Δ is parallel in $\mathbb{R}^N$ along the leaves of D.*

Proof: It suffices to show that the orthogonal complement $\Gamma \oplus P$ of Δ is parallel in $\mathbb{R}^N$ along the leaves of D. First observe that

$$\Gamma \oplus P = \text{span}\{\tilde{\nabla}_X \mu : X \in TM \text{ and } \mu \in P\}.$$

Then, we have from (5) that

$$\tilde{\nabla}_Y \tilde{\nabla}_X \mu = \tilde{\nabla}_X \tilde{\nabla}_Y \mu + \tilde{\nabla}_{[Y,X]} \mu \in \Gamma \oplus P$$

for any $\mu \in P$, $Y \in D$ and $X \in TM$, and the proof follows. ∎

Define $F \colon N^{n+r} \to \mathbb{R}^N$ as the restriction of the map

$$\lambda \in \Lambda \mapsto f(\pi(\lambda)) + \lambda$$

to a tubular neighborhood N^{n+r} of the 0-section $j \colon M^n \hookrightarrow N^{n+r}$ of Λ where it is an immersion. Hence $f = F \circ j$ and

$$T_{j(x)} N = j_* T_x M \oplus \Lambda(x) \tag{6}$$

for any $x \in M^n$.

By Lemma 7 we have that F is Δ-ruled where $\Delta(\lambda) = \Delta(\pi(\lambda))$. From

$$\langle \tilde{\nabla}_X \lambda, \mu \rangle = -\langle \lambda, \tilde{\nabla}_X \mu \rangle = 0$$

for $\lambda \in \Lambda$, $\mu \in P$ and $X \in TM$, it follows that $\mathcal{P} \subset N_F N$ where $\mathcal{P}(\lambda) = P(\pi(\lambda))$. Moreover,

$$\Delta = \mathcal{N}(\alpha_{\mathcal{P}}^F).$$

In fact, the inclusion $\Delta \subset \mathcal{N}(\alpha_{\mathcal{P}}^F)$ holds because $\mathcal{P}$ is constant along Δ. For the opposite inclusion observe that $\alpha_{\mathcal{P}}^F|_{TM \times TM} = \alpha_P$. We easily obtain from (6) that equality is satisfied along M^n. To conclude the proof observe that the dimension of $\mathcal{N}(\alpha_{\mathcal{P}}^F)$ can only decrease along $\Delta \subset N^{n+r}$ from its value on M^n if N^{n+r} is taken small enough.

We summarize the above facts in the following statement.

Proposition 8. *The immersion $F \colon N^{n+r} \to \mathbb{R}^N$ is a Δ-ruled extension of $f \colon M^n \to \mathbb{R}^N$ with $D^d(x) = \Delta^{d+r}(x) \cap T_x M$ at any $x \in M^n$. Moreover, there is an orthogonal splitting*

$$N_F N = \mathcal{L} \oplus \mathcal{P}$$

where rank $\mathcal{L} = \ell - r$ such that $\Delta = \mathcal{N}(\alpha_{\mathcal{P}}^F)$ and $\mathcal{P}$ is constant along Δ.

We see from (4) that there are two extreme cases to consider.

Corollary 9. *In the situation of Proposition 8, we have:*

(i) If $k = n - d + \ell$ ($r = 0$) then f is D-ruled.

(ii) If $k = n - d$ ($r = \ell$) then Δ is contained in the relative nullity distribution of F.

3   The proof

A key ingredient in the proof of Theorem 1 is the basic property of regular elements of a bilinear form observed by Moore [6] that we state below.

Let $\beta\colon V \times U \to W$ be a bilinear form between finite dimensional real vector spaces. We call $Z \in V$ a (left) *regular element* of β if the map $\beta_Z = \beta(Z, \cdot)\colon U \to W$ satisfies

$$\dim \beta_Z(U) = \max\{\dim \beta_Y(U) : Y \in V\},$$

and denote by $RE(\beta)$ the subset of regular elements of β. It is a well-known fact that the set $RE(\beta)$ is open and dense in V.

Proposition 10. *If $\beta\colon V \times U \to W$ is a bilinear form and $Z \in RE(\beta)$, then*

$$\beta(V, \ker \beta_Z) \subset \beta_Z(U).$$

With the notations from Section 1, consider a 1-regular locally substantial isometric immersion $f\colon M^n \to \mathbb{R}^N$ such that $s(x)$ has a constant value $0 < s < n$. Let $\mu_1 \in RE(\phi)$ be a globally defined unit vector field and set $\phi_{\mu_1} = \phi(\mu_1, \cdot)$. We also assume that the subspaces $S_1(x) \subset \mathcal{S}(x)$ defined by

$$S_1(x) = \phi_{\mu_1}(T_x M)$$

have constant dimension $1 \le s_1 \le s$. Hence, the tangent subspaces

$$D_1(x) = \ker \phi_{\mu_1}(x)$$

satisfy $\dim D_1(x) = n - s_1$.

Lemma 11. *Suppose that $s \le 6$. Then $D = \mathcal{N}(\phi)$ satisfies*

$$\dim D \ge n - s. \tag{7}$$

Proof: The assertion is clearly true if $s_1 = s$. If $s_1 < s$, consider the orthogonal splitting

$$\mathcal{S} = S_1 \oplus S_1^\perp$$

and let $\psi\colon N_1^\perp \oplus TM \to S_1^\perp$ denote the bilinear form defined by

$$\psi(\mu, X) = (\nabla_X^\perp \mu)_{S_1^\perp}.$$

Take $\mu_2 \in RE(\phi) \cap RE(\psi)$ and set $t = \dim \psi(\mu_2, TM)$. Then $S_2 = \phi_{\mu_2}(TM)$ satisfies

$$\dim (S_1 + S_2) = s_1 + t \quad \text{and} \quad \dim S_1 \cap S_2 = s_1 - t.$$

It follows using Proposition 10 that

$$\dim D_1 \cap D_2 \ge \dim D_1 - \dim S_1 \cap S_2 \ge n - 2s_1 + t. \tag{8}$$

If $t = s_1$ then $S_1 \cap S_2 = 0$. Thus $D_1 = D_2$. In particular (7) holds if $s_1 = 1$ since this forces $t = 1$. Therefore, we may assume

$$s_1 \ge 2. \tag{9}$$

We first analyze the case $t = 1$. Then $H = \ker \psi(\mu_2, \cdot)$ is a hyperplane in TM. The Codazzi equation gives

$$A_{\nabla_X^\perp \delta} Y = A_{\nabla_Y^\perp \delta} X \tag{10}$$

for any $\delta \in N_1^\perp$. Hence

$$A_{\nabla_Z^\perp \mu_2} X = A_{\nabla_X^\perp \mu_2} Z = 0$$

for any $Z \in D_1$ and $X \in H$. This implies that $\dim \phi_{\mu_2}(D_1) \le 1$. Otherwise, there would exist a two-dimensional plane in S_1 such that the corresponding shape operators would have the same kernel of codimension one. But then a vector in this plane would belong to $N_1^\perp$, and this is a contradiction. It follows that $\dim D_1 \cap D_2 \ge n - s_1 - 1$.

If $\mathcal{S} = S_1 + S_2$ then (7) holds since $s = s_1 + 1$ and $D = D_1 \cap D_2$. If otherwise, we just repeat the process and obtain subspaces $S_1, \ldots, S_m$ and $D_1, \ldots, D_m$, $m = s - s_1 + 1$, such that $\mathcal{S} = S_1 + \cdots + S_m$ and $\dim D_1 \cap \cdots \cap D_m \ge n - s_1 - m + 1 - s$. Then $D = D_1 \cap \cdots \cap D_m$, and (7) follows.

By the above, we may assume

$$t \ge 2. \tag{11}$$

We argue for the case $s = 6$, the other cases being similar and easier. If $t = s_1$ then $s_1 = 2, 3$. In these cases we have seen that $D_1 = D_2$, and thus (7) holds. Hence, in view of (9) and (11) we may assume that

$$s_1 > t \ge 2.$$

Thus, it remains to consider the cases $(s_1, t) = (3, 2)$ and $(s_1, t) = (4, 2)$. In the latter case, we have that $\mathcal{S} = S_1 + S_2$, and (7) follows from (8). In the first case, we have $\dim(S_1 + S_2) = 5$, $\dim S_1 \cap S_2 = 1$ and $\dim D_1 \cap D_2 \ge n - 4$. We now repeat the process and obtain S_3 such that $\mathcal{S} = S_1 + S_2 + S_3$ and $\dim S_i \cap S_j = 1$ if $i \ne j$. In this case, it is now clear that $\dim D \ge n - 5$. ∎

Remark 12. Our proof does not work for $s = 7$. In fact, in this case we may have $s_1 = 5$ and $t = 2$. Thus $\mathcal{S} = S_1 + S_2$ and (8) only yields $\dim D \ge n - 8$.

Consider the global smooth orthogonal splitting $N_1^f = L^{p-s} \oplus \mathcal{S}^s$. Then, we have the global orthogonal splitting

$$N_f M = L^{p-s} \oplus P \tag{12}$$

where $P = \mathcal{S}^s \oplus N_1^\perp$.

Lemma 13. *The following facts hold:*

(i) $D = \mathcal{N}(\alpha_{\mathcal{S}})$.

(ii) P *is parallel along D in the normal connection.*

Proof: To prove (i) it suffices to show that

$$D_1 = \mathcal{N}(\alpha_{S_1}), \tag{13}$$

i.e., that $Y \in D_1$ if and only if $A_{\nabla_X^\perp \mu_1} Y = 0$ for any $X \in TM$. But this follows from (10).

For the proof of (ii) observe that

$$\nabla_Y^\perp \nabla_X^\perp \mu_1 - \nabla_X^\perp \nabla_Y^\perp \mu_1 - \nabla_{[Y,X]}^\perp \mu_1 = 0$$

by the Ricci equation. Take $Y \in D_1$ and $X \in TM$. Then,

$$\nabla_Y^\perp (\nabla_X^\perp \mu_1)_{S_1} + \nabla_Y^\perp (\nabla_X^\perp \mu_1)_{N_1^\perp} = \nabla_X^\perp \nabla_Y^\perp \mu_1 + \nabla_{[Y,X]}^\perp \mu_1 \in P.$$

By Proposition 10, the second term in the left-hand-side belongs to P. It follows that $\nabla_Y^\perp \delta \in P$ for any $Y \in D_1$ and $\delta \in S_1$. This easily gives (ii). ∎

Proof of Theorem 1. Assume first that $s = p$, that is, that $\mathcal{S} = N_1^f$. Then, Lemma 11 and part (i) of Lemma 13 imply that $\nu_f \geq n - p$.

Suppose now that $s < p$. Let $M_0 \subset M^n$ be an open dense subset of M^n where all previously defined vector subspaces have constant dimensions, and thus form vector bundles. Define U_k as the union of all connected components of M_0 in which the subspaces $\Gamma(x)$ given by (3) with respect to the splitting (12) have the same value k. In view of Lemma 11 and Lemma 13, we can apply Proposition 8 for $f|_{U_k}$, and with the notations therein we have

$$d \geq n - s, \quad \ell = p - s, \quad r = p - k \quad \text{and} \quad m + p - k.$$

Then,

$$\text{rank } \mathcal{L} = k - s \quad \text{and hence} \quad \text{rank } N_1^F = k.$$

Also,

$$\text{rank } \Delta = d + p - k \geq n - s + p - k = m - s.$$

From (4) we have $s \leq k \leq p$. If $k = p$, then f is D-ruled by part (i) of Corollary 9. Moreover, if $k = s$ then the rulings Δ of its extension F are contained in the relative nullity distribution of F by part (ii) of the same result.

The global assertion in (ii) for the case $1 = s < p$ is due to the fact that $s = 1$ implies $k = 1$, as follows from (2). It is also a consequence of (2) that $k \leq 4$ if $s = 2$, hence in this case $U_k = \emptyset$ for $k = 5, 6$. ∎

4 Examples

In this section we give examples of Euclidean submanifolds satisfying the conditions in part $(iii) - (a)$ of Theorem 1. More precisely, we construct ruled submanifolds M^{2m} in $\mathbb{R}^{2m+6}$ with four dimensional first normal bundle such that $\mathcal{S}$ has rank two and is constant along the codimensional two rulings. These examples show that the result cannot be improved since the rulings are not in the relative nullity distribution and their dimension achieve the minimum possible value given by our estimate.

Let $g\colon L^2 \to \mathbb{R}^{2(m+3)}$, $m \geq 2$, be a substantial elliptic surface in the sense of [2], i.e., there exists a (unique up to sign) almost complex structure J on L^2 such that

$$\alpha_g(Z, Z) + \alpha_g(JZ, JZ) = 0$$

for any $Z \in TL$. For instance, the surface can be minimal, which is equivalent to J being orthogonal. Then, it turns out that the normal bundle of g splits orthogonally as

$$N_g L = N_1^g \oplus \cdots \oplus N_{m+2}^g$$

where each plane bundle N_k^g, $1 \leq k \leq m+2$, is its kth-normal bundle; see [2] for details. Recall that the kth-normal space N_k^h, $k \geq 2$, of an isometric immersion $h\colon M^n \to \mathbb{R}^N$ at $x \in M^n$ is defined as

$$N_k^h(x) = \mathrm{span}\{\alpha_h^{k+1}(X_1, \ldots, X_{k+1}) \, : \, X_1, \ldots, X_{k+1} \in T_x M\},$$

where $\alpha_h^\ell\colon TM \times \cdots \times TM \to N_h M$, $\ell \geq 3$, is the ℓth-fundamental form given by

$$\alpha_h^\ell(X_1, \ldots, X_\ell) = \pi^{\ell-1}(\nabla_{X_\ell}^\perp \cdots \nabla_{X_3}^\perp \alpha(X_2, X_1)).$$

Here π^ℓ is the orthogonal projection onto $(N_1^h \oplus \ldots \oplus N_{\ell-1}^h)^\perp \cap N_h M$.

Define $f\colon M^{2m} \to \mathbb{R}^{2(m+3)}$ as the restriction of the map

$$\xi \in N_1^g \oplus \cdots \oplus N_{m-1}^g \mapsto g(\pi(\xi)) + \xi$$

to a tubular neighborhood of the 0-section L^2 of $\pi\colon N_1^g \oplus \cdots \oplus N_{m-1}^g \to L^2$ where it is an immersion. Given $\xi \in M^{2m} \setminus L^2$, we claim that

$$f_* T_\xi M \oplus N_1^f(\xi) = g_* T_x L \oplus N_1^g(x) \oplus \cdots \oplus N_{m+1}^g(x), \quad x = \pi(\xi).$$

Let $\tilde{\xi}$ be a local section of $N_1^g \oplus \cdots \oplus N_{m-1}^g$ on a neighbourhood U of x such that $\tilde{\xi}(U) \subset M^{2m}$ and $\tilde{\xi}(x) = \xi$. Then

$$f_* \tilde{\xi}_* X = g_* X + \tilde{\nabla}_X \tilde{\xi} \tag{14}$$

for any $X \in T_x L$. On the other hand, for a vertical vector $V \in T_\xi M$ we have

$$f_* V = V.$$

Hence $N_1^g(x) \oplus \cdots \oplus N_{m-1}^g(x) \subset f_* T_\xi M$ and $f_* T_\xi M \subset g_* T_x L \oplus N_1^g(x) \oplus \cdots \oplus N_m^g(x)$. Regarding the local section $\tilde{\xi}$ as a vertical vector field of M^{2m}, we obtain

$$\tilde{\nabla}_X \tilde{\xi} = \tilde{\nabla}_{\tilde{\xi}_* X} f_* \tilde{\xi} \in f_* T_\xi M \oplus N_1^f(\xi). \tag{15}$$

Thus $N_m^g(x) \subset f_* T_\xi M \oplus N_1^f(\xi)$, hence also $g_* T_x L \subset f_* T_\xi M \oplus N_1^f(\xi)$ by (14). Differentiating (14) yields

$$\tilde{\nabla}_{\tilde{\xi}_* Y} f_* \tilde{\xi}_* \tilde{X} = \tilde{\nabla}_Y g_* \tilde{X} + \tilde{\nabla}_Y \tilde{\nabla}_X \tilde{\xi}$$

for all $X, Y \in T_x L$, where $\tilde{X}$ is any vector field on a neighbourhood of x with $\tilde{X}(x) = X$. Thus $N_1^f(\xi) \subset g_* T_x L \oplus N_m^g(x) \oplus N_{m+1}^g(x)$ and $N_{m+1}^g(x) \subset N_1^f(\xi)$, and the claim follows.

Note also that the rulings of f are not in its relative nullity distribution. In fact, it follows from (15) that

$$\mathrm{span}\{\alpha_f(Z, V) : Z, V \in T_\xi M \text{ and } V \text{ vertical}\} = (g_* T_x L \oplus N_m^g(x)) \cap N_1^f(\xi). \tag{16}$$

We have from the claim that $N_f M = N_1^f \oplus N_{m+2}^g$. Thus, the immersion f is ruled by $N_1^g \oplus \cdots \oplus N_{m-1}^g$ and $\mathcal{S} = N_{m+1}^g$ has rank two and is constant in the ambient space along the rulings. Moreover, by (16) the rulings are not in the relative nullity distribution and their dimension satisfy the equality in the estimate given in part $(iii)-(a)$ of Theorem 1.

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IMPA – Estrada Dona Castorina, 110
22460-320 – Rio de Janeiro – Brazil
E-mail: marcos@impa.br

Universidade Federal de São Carlos
13565-905 – São Carlos – Brazil
E-mail: tojeiro@dm.ufscar.br